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FINITE ELEMENT SOLUTION FOR AXISYMMETRICAL SHELLS

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A comprehensive method of analysis of axisymmetrical thin elastic shells of revolution based on finite element approach is presented in this paper. The basic finite element, by means of which any axisymmetric shell may be approximated, is a truncated conical ring. In the limiting case such an element is replaced by a short cylinder; at the ends of a shell, shallow spherical caps are employed. By using many elements, thickness variation of the shell can be approximated. The procedures are stated in matrix algebra and in principle are based on the "displacement method" of analysis. An example illustrates convergence of the proposed method.

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Introduction

Axisymmetrical thin shells of revolution are widely used in flight and other structures. These shells often have arbitrary shape and thickness variations to meet functional and manufacturing requirements. For various reasons, such as junctures with adjoining parts, discontinuity in load distribution, etc., important bending stresses are developed in the shells in addition to membrane stresses. The analysis of such shells is, therefore, of great importance to design engineers.

A review of literature shows that mathematical solutions of bending problems of axisymmetrical shells of revolution are available only for a few special cases. The governing differential equations originally formulated by H. Reissner and Meissner based on the classical theory of elasticity have been solved exactly for spheres, cones and cylinders of constant thickness(1)(2). For shells with variable thickness only a few special solutions exist(1)(3).

The problem becomes more complicated when the deformations are large. E Reissner formulated the governing differential equations of the "small finite deflection theory" which considers small deformations but arbitrary rotations(4). These equations, in general, can be solved only by asymptotic integration, and solutions can be found for a few simple cases(5).

The governing differential equations, however, can be solved by numerical procedures, i.e., by finite difference method with the aid of a digital computer. Such solutions have been formulated by Sepetoski, Pearson, Dingwell, Adkins(9), and Soare(11). However, one is not assured of a high degree of accuracy even when a fine mesh is used.

This paper describes a finite element solution of thin shells of

Nomenclature

A	=	force transformation matrix
a	=	radius of sphere or cylinder
B	=	displacement transformation matrix
E	=	modulus of elasticity
f	=	element flexibility matrix
H, M, N	=	stress-resultants
i, j	=	designations for upper and lower edges of an element
K	=	stiffness matrix of the structure (assemblage)
$k, \bar{k}$	=	element stiffness matrix
ℓ	=	element length; also $\ell = (at)^{\frac{1}{2}} / \left[12(1-\nu^2)\right]^{\frac{1}{4}}$ for spherical cap
R	=	joint or edge force matrix
r	=	displacement matrix; horizontal radius
$\bar{r}$	=	ratio s_j/s_i
S	=	element force matrix
s	=	distance along cone from apex
	–	thickness of shell
U	=	unknown joint displacement matrix
v	=	element displacement matrix; meridianal displacements
X	=	unknown joint force matrix
y	=	$2\left[3(1-\nu^2)\right]^{\frac{1}{4}} (2 \tan \alpha/t)^{\frac{1}{2}} (s)^{\frac{1}{2}}$
α	=	inclination of conical element
$\Delta, \theta; \delta, \chi$	=	displacements; deformations
κ	=	$\left[3(1-\nu^2)\, a^2/t^2\right]^{\frac{1}{4}}$
ν	=	Poisson's ratio

Other symbols and super- and subscripts are defined in the text or on the figure.

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revolution in which the actual shell of arbitrary shape and smoothly varying thickness is approximated by a series of truncated cones and cylinders and shallow spherical caps for the end pieces. With the structural properties of the elements known from available solutions of conical, cylindrical, and spherical shells, a solution of the general problem can be obtained for any axisymmetrically applied loads by the matrix method of analysis familiar to structural engineers. If the loads are applied in several increments and at each step the deformed shape of the shell is considered as the new outline of the shell, the solution becomes equivalent to one based on a large deformation theory. Also after each step of loading the stresses and strains in the individual elements determine whether or not a new set of constants defining material properties should be assigned in the next load increment. In this manner, the method can be extended to include non-elastic material properties provided the biaxial stress-strain relation of the material is known.

In this paper only the basic principles and formulations necessary for a small deformation elastic analysis are presented. Readers interested in further details of the method are referred to a technical report by the authors[10].

Conical Elements

In recent years the basic principles of the finite element method of structural analysis have become well known and are adequately described in the literature[6]. Here to solve the problem, the "displacement method" of analysis is employed. This solution procedure is first discussed with reference to the conical elements. Later, the needed additional equations for cylindrical elements and the spherical cap are given.

In a typical n-th conical element such as shown in Fig. 1, $M_i^{(n)}$, $M_j^{(n)}$, $H_i^{(n)}$, and $H_j^{(n)}$ are self-equilibrating stress-resultants and may be applied independently. For vertical equilibrium $N_j^{(n)}$, however, must be balanced by $N_i^{(n)}$; hence, $N_i^{(n)} = \bar{r}\, N_j^{(n)}$ where $\bar{r} = s_j/s_i$. Together these five independent element edge-forces acting on a truncated cone form the $S^{(n)}$ column matrix. As may be seen from Fig. 1, the rotations $\chi_i^{(n)}$, $\chi_j^{(n)}$, the displacements $\delta_i^{(n)}$, $\delta_j^{(n)}$ and the stretch $e^{(n)}$ in the direction of the cone generator occur corresponding to this system of forces. Positive sense for these five displacements is in the positive direction of the applied forces and for any n-th conical element, they can be related to the applied edge forces through a flexibility matrix in the following manner:

$$\begin{Bmatrix} \chi_i^{(n)} \\ \chi_j^{(n)} \\ \delta_i^{(n)} \\ \delta_j^{(n)} \\ e^{(n)} \end{Bmatrix} = \begin{bmatrix} f_{11}^{(n)} & f_{12}^{(n)} & f_{13}^{(n)} & f_{14}^{(n)} & f_{15}^{(n)} \\ f_{21}^{(n)} & f_{22}^{(n)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & f_{33}^{(n)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & f_{44}^{(n)} & \cdot \\ f_{51}^{(n)} & \cdot & \cdot & \cdot & f_{55}^{(n)} \end{bmatrix} \begin{Bmatrix} M_i^{(n)} \\ M_j^{(n)} \\ H_i^{(n)} \\ H_j^{(n)} \\ N_j^{(n)} \end{Bmatrix} \tag{1}$$

or simply as

$$\left\{ v^{(n)} \right\} = \left[f^{(n)} \right] \left\{ S^{(n)} \right\} \tag{2}$$

After lengthy manipulations, based on the bending theory of conical shells[1], together with the membrane solution for stress-resultants N_i and N_j, the following relations corresponding to Eq. 1 written in expanded form are obtained: *

$$\begin{aligned}
\chi_i = -Cp1 \Big\{ & \left[a_{12}b_2(y_i) - a_{22}b_1(y_i) + b_{12}k_2(y_i) - b_{22}k_1(y_i)\right] M_i \\
& + \left[a_{14}b_2(y_i) - a_{24}b_1(y_i) + b_{14}k_2(y_i) - b_{24}k_1(y_i)\right] M_j \\
& + \left[a_{11}b_2(y_i) - a_{21}b_1(y_i) + b_{11}k_2(y_i) - b_{21}k_1(y_i)\right] H_i \\
& + \left[a_{13}b_2(y_i) - a_{23}b_1(y_i) + b_{13}k_2(y_i) - b_{23}k_1(y_i)\right] H_j \Big\} \\
& + (\bar{r} \cot \alpha / Et) N_j
\end{aligned}$$

$$\begin{aligned}
\chi_j = Cp1 \Big\{ & \left[a_{12}b_2(y_j) - a_{22}b_1(y_j) + b_{12}k_2(y_j) - b_{22}k_1(y_j)\right] M_i \\
& + \left[a_{14}b_2(y_j) - a_{24}b_1(y_j) + b_{14}k_2(y_j) - b_{24}k_1(y_j)\right] M_j \\
& + \left[a_{11}b_2(y_j) - a_{21}b_1(y_j) + b_{11}k_2(y_j) - b_{21}k_1(y_j)\right] H_i \\
& + \left[a_{13}b_2(y_j) - a_{23}b_1(y_j) + b_{13}k_2(y_j) - b_{23}k_1(y_j)\right] H_j \Big\} \\
& - (\cot \alpha / Et) N_j
\end{aligned}$$

$$\begin{aligned}
\delta_i = -Cp3 \Big\{ & \left[a_{12}b_5(y_i) + a_{22}b_6(y_i) + b_{12}k_5(y_i) + b_{22}k_6(y_i)\right] M_i \\
& + \left[a_{14}b_5(y_i) + a_{24}b_6(y_i) + b_{14}k_5(y_i) + b_{24}k_6(y_i)\right] M_j \\
& + \left[a_{11}b_5(y_i) + a_{21}b_6(y_i) + b_{11}k_5(y_i) + b_{21}k_6(y_i)\right] H_i \\
& + \left[a_{13}b_5(y_i) + a_{23}b_6(y_i) + b_{13}k_5(y_i) + b_{23}k_6(y_i)\right] H_j \Big\} \\
& + (\nu\, s_j \cos \alpha / Et) N_j
\end{aligned}$$

---

*For simplicity, superscripts (n) on all displacements and stress-resultants, and subscripts on α are omitted.

$$
\begin{aligned}
\delta_j = Cp3 \Big\{ & \Big[a_{12} b_5(y_j) + a_{22} b_6(y_j) + b_{12} k_5(y_j) + b_{22} k_6(y_j) \Big] M_i \\
& + \Big[a_{14} b_5(y_j) + a_{24} b_6(y_j) + b_{14} k_5(y_j) + b_{24} k_6(y_j) \Big] M_j \\
& + \Big[a_{11} b_5(y_j) + a_{21} b_6(y_j) + b_{11} k_5(y_j) + b_{21} k_6(y_j) \Big] H_i \\
& + \Big[a_{13} b_5(y_j) + a_{23} b_6(y_j) + b_{13} k_5(y_j) + b_{23} k_6(y_j) \Big] H_j \Big\} \\
& - (\nu\, s_j \cos\alpha / Et) N_j
\end{aligned}
$$

See Note below

$$
\begin{aligned}
e = Cp2 \Big\{ & \Big[a_{12} \big\langle b_9(y_j) - b_9(y_i) \big\rangle + a_{22} \big\langle b_{10}(y_j) - b_{10}(y_i) \big\rangle \\
& \quad + b_{12} \big\langle k_9(y_j) - k_9(y_i) \big\rangle + b_{22} \big\langle k_{10}(y_j) - k_{10}(y_i) \big\rangle \Big] M_i \\
& + \Big[a_{14} \big\langle b_9(y_j) - b_9(y_i) \big\rangle + a_{24} \big\langle b_{10}(y_j) - b_{10}(y_i) \big\rangle \\
& \quad + b_{14} \big\langle k_9(y_j) - k_9(y_i) \big\rangle + b_{24} \big\langle k_{10}(y_j) - k_{10}(y_i) \big\rangle \Big] M_j \qquad (3) \\
& + \Big[a_{11} \big\langle b_9(y_j) - b_9(y_i) \big\rangle + a_{21} \big\langle b_{10}(y_j) - b_{10}(y_i) \big\rangle \\
& \quad + b_{11} \big\langle k_9(y_i) - k_9(y_i) \big\rangle + b_{21} \big\langle k_{10}(y_j) - k_{10}(y_i) \big\rangle \Big] H_i \\
& + \Big[a_{13} \big\langle b_9(y_j) - b_9(y_i) \big\rangle + a_{23} \big\langle b_{10}(y_j) - b_{10}(y_i) \big\rangle \\
& \quad + b_{13} \big\langle k_9(y_j) - k_9(y_i) \big\rangle + b_{23} \big\langle k_{10}(y_j) - k_{10}(y_i) \big\rangle \Big] H_j \Big\} \\
& + (s_j \log \bar{r} / Et) N_j
\end{aligned}
$$

where $Cp1 = 2 \left[3(1-\nu^2) \right]^{\frac{1}{2}} \cot\alpha / Et^2$, $Cp2 = \cot\alpha / Et$,

$Cp3 = \cot\alpha \cos\alpha / Et$

Note to Printer:
⟨ ⟩'s are used in lieu of ordinary large ( )'s. Replace with large ( ) if available.

$a_{11} = -\sin\alpha\ s_i\ d_{11} / \Delta$ $\qquad a_{12} = -\ y_i^2\ d_{12} / 2\Delta$

$a_{13} = -\sin\alpha\ s_j\ d_{13} / \Delta$ $\qquad a_{14} = -\ y_j^2\ d_{14} / 2\Delta$

$a_{21} = \sin\alpha\ s_i\ d_{21} / \Delta$ $\qquad a_{22} = y_i^2\ d_{22} / 2\Delta$

$a_{23} = \sin\alpha\ s_j\ d_{23} / \Delta$ $\qquad a_{24} = y_j^2\ d_{24} / 2\Delta$

$b_{11} = -\sin\alpha\ s_i\ d_{31} / \Delta$ $\qquad b_{12} = -\ y_i^2\ d_{32} / 2\Delta$

$b_{13} = -\sin\alpha\ s_j\ d_{33} / \Delta$ $\qquad b_{14} = -\ y_j^2\ d_{34} / 2\Delta$

$b_{21} = \sin\alpha\ s_i\ d_{41} / \Delta$ $\qquad b_{22} = y_i^2\ d_{42} / 2\Delta$

$b_{23} = \sin\alpha\ s_j\ d_{43} / \Delta$ $\qquad b_{24} = y_j^2\ d_{44} / 2\Delta$

$$\Delta = \begin{vmatrix} b_1(y_i) & b_2(y_i) & k_1(y_i) & k_2(y_i) \\ b_4(y_i) & -b_3(y_i) & k_4(y_i) & -k_3(y_i) \\ b_1(y_j) & b_2(y_j) & k_1(y_j) & k_2(y_j) \\ b_4(y_j) & -b_3(y_j) & k_4(y_j) & -k_3(y_j) \end{vmatrix}$$

d_{ji} = minors of item ij in the determinant Δ.

$b_1(y) = \text{ber } y - 2y^{-1}\text{bei}'y$ $\qquad b_2(y) = \text{bei } y + 2y^{-1}\text{ber}'y$

$k_1(y) = \text{ker } y - 2y^{-1}\text{kei}'y$ $\qquad k_2(y) = \text{kei } y + 2y^{-1}\text{ker}'y$

$b_3(y) = y\ \text{ber}'y - 2(1-\nu)\ b_1(y)$ $\qquad b_4(y) = y\ \text{bei}'y - 2(1-\nu)\ b_2(y)$

$k_3(y) = y\ \text{ker}'y - 2(1-\nu)\ k_1(y)$ $\qquad k_4(y) = y\ \text{kei}'y - 2(1-\nu)\ k_2(y)$

$b_5(y) = -\frac{1}{2}\left[y\ \text{ber}'y - 2(1+\nu)\ b_1(y)\right]$ $\qquad b_6(y) = -\frac{1}{2}\left[y\ \text{bei}'y - 2(1+\nu)\ b_2(y)\right]$

$k_5(y) = -\frac{1}{2}\left[y\ \text{ker}'y - 2(1+\nu)\ k_1(y)\right]$ $\qquad k_6(y) = -\frac{1}{2}\left[y\ \text{kei}'y - 2(1+\nu)k_2(y)\right]$

$b_9(y) = \nu\ \text{ber } y - 2(1+\nu)\ y^{-1}\text{bei}'y$ $\qquad b_{10}(y) = \nu\ \text{bei } y + 2(1+\nu)\ y^{-1}\text{ber}'y$

$k_9(y) = \nu\ \text{ker } y - 2(1+\nu)\ y^{-1}\text{kei}'y$ $\qquad k_{10}(y) = \nu\ \text{kei } y + 2(1+\nu)\ y^{-1}\text{ker}'y$

The appropriate terms in Eq. 3 define the elements of the matrix $f^{(n)}$ in Eq. 1. Some elements in this matrix bear a ratio of $\bar{r}$ to the corresponding quantities on the opposite side of the diagonal and the matrix is not symmetrical. Nevertheless, it can be shown that Betti's law is satisfied, and the matrix could be symmetrized by using in the solution total force quantities around each edge. However, it appears more advantageous to work with edge forces per unit of length since these are the quantities used in the shell equations.

The accuracy of the matrix $f^{(n)}$, judged from the values of the quantities on opposite sides of the main diagonal, has been investigated over a wide range of thickness, lengths, angles and radii of cones. It was found that the upper left 4 by 4 portion of the matrix is very good in almost every case. However, discrepancies on opposite sides of the diagonal develop as α approaches 90°. Nevertheless, the matrix remains valid as long as $\tan \alpha$ is finite. The fifth column and fifth row which depend on different mathematical functions diverge as the geometry of the cone approaches extreme cases. To eliminate discrepancies between quantities of the fifth column and the fifth row, it is recommended that the quantities of the fifth row for flat conical elements be used to establish those of the fifth column since for a small α plate action predominates. For nearly vertical conical elements, quantities of the fifth column should be used to establish those of the fifth row. With minor modifications the basic procedure is applicable for conical rings with larger periphery on the top than on the bottom.

For subsequent work, through a displacement transformation matrix B, the five displacements shown earlier in Fig. 1 must be related to the six possible edge displacements. Positive directions for the six new displacements are defined in Fig. 2 to agree with later usage. This transformation is

$$\begin{Bmatrix} \chi_i^{(n)} \\ \chi_j^{(n)} \\ \delta_i^{(n)} \\ \delta_j^{(n)} \\ e^{(n)} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\cos\alpha_n & \sin\alpha_n & 0 & \cos\alpha_n & -\sin\alpha_n \end{array}\right] \left\{\begin{array}{c} \theta_i^{(n)} \\ \Delta_{hi}^{(n)} \\ \Delta_{vi}^{(n)} \\ \hline \theta_j^{(n)} \\ \Delta_{hj}^{(n)} \\ \Delta_{vj}^{(n)} \end{array}\right\} \tag{4}$$

or in alternative symbolic forms, one has

$$\left\{v^{(n)}\right\} = \left[B^{(n)}\right]\left\{r^{(n)}\right\} = \left[B_i^{(n)} \,\vdots\, B_j^{(n)}\right] \left\{\begin{array}{c} r_i^{(n)} \\ \hline r_j^{(n)} \end{array}\right\} \tag{5}$$

In an analogous manner, using the force transformation matrix A, the five independent stress-resultants shown in Fig. 1 can be related to the six edge forces shown in Fig. 2. With the aid of these diagrams, one obtains

$$\left\{\begin{array}{c} T_i^{(n)} \\ P_{hi}^{(n)} \\ P_{vi}^{(n)} \\ \hline T_j^{(n)} \\ P_{hj}^{(n)} \\ P_{vj}^{(n)} \end{array}\right\} = \left[\begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -\bar{r}^{(n)}\cos\alpha_n \\ 0 & 0 & 0 & 0 & \bar{r}^{(n)}\sin\alpha_n \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cos\alpha_n \\ 0 & 0 & 0 & 0 & -\sin\alpha_n \end{array}\right] \begin{Bmatrix} M_i^{(n)} \\ M_j^{(n)} \\ H_i^{(n)} \\ H_j^{(n)} \\ N_j^{(n)} \end{Bmatrix} \tag{6}$$

or in abbreviated forms,

$$\left\{R^{(n)}\right\} = \left\{\begin{array}{c} R_i^{(n)} \\ \hdashline R_j^{(n)} \end{array}\right\} = \left[A^{(n)}\right]\left\{S^{(n)}\right\} = \left[\begin{array}{c} A_i^{(n)} \\ \hdashline A_j^{(n)} \end{array}\right]\left\{S^{(n)}\right\} \tag{7}$$

Structural Stiffness of Assemblage

After the various structural properties for a truncated conical element have been established, structural stiffness for an assemblage consisting of conical elements can be formulated. To accomplish this, from Eq. 2 one solves for $S^{(n)}$ and substitutes into Eq.7. Whence upon combining this result with Eq. 5, and noting that $k^{(n)} = \left[f^{(n)}\right]^{-1}$, one obtains

$$R^{(n)} = A^{(n)}\, k^{(n)}\, B^{(n)}\, r^{(n)} \tag{8}$$

or

$$R^{(n)} = \bar{k}^{(n)}\, r^{(n)} \quad \text{where} \quad \bar{k}^{(n)} = A^{(n)}\, k^{(n)}\, B^{(n)} \tag{9}$$

By noting from Eqs. 5 and 7 that $A^{(n)}$ and $B^{(n)}$ can be written in partitioned form, it is possible to recast the expression for $\bar{k}^{(n)}$ also into a partitioned form as

$$\bar{k}^{(n)} = \left[\begin{array}{c} A_i^{(n)} \\ \hdashline A_j^{(n)} \end{array}\right] k^{(n)} \left[\begin{array}{c:c} B_i^{(n)} & B_j^{(n)} \end{array}\right] = \left[\begin{array}{c:c} \bar{k}_{ii}^{(n)} & \bar{k}_{ij}^{(n)} \\ \hdashline k_{ji}^{(n)} & k_{jj}^{(n)} \end{array}\right] \tag{10}$$

where

$$\begin{array}{ll} \bar{k}_{ii}^{(n)} = A_i^{(n)}\, k^{(n)}\, B_i^{(n)} & \bar{k}_{ij}^{(n)} = A_i^{(n)}\, k^{(n)}\, B_j^{(n)} \\ \bar{k}_{ji}^{(n)} = A_j^{(n)}\, k^{(n)}\, B_i^{(n)} & \bar{k}_{jj}^{(n)} = A_j^{(n)}\, k^{(n)}\, B_j^{(n)} \end{array} \tag{11}$$

It should be noted that an i-th edge of the n-th element and the j-th edge of the (n-1)-th element meet at the n-th <u>joint</u> of the assemblage. Therefore, the <u>joint force</u>, Fig. 3, is

$$R_n = \begin{Bmatrix} (T_i^{(n)} + T_j^{(n-1)}) \\ (P_{hi}^{(n)} + P_{hj}^{(n-1)}) \\ (P_{vi}^{(n)} + P_{vj}^{(n-1)}) \end{Bmatrix} = R_i^{(n)} + R_j^{(n-1)} \tag{11}$$

For the same n-th joint of the assemblage, the displacement r_n is common to the displacement of the bottom edge of the upper element and to the upper edge of the lower element, i.e.

$$r_n = r_i^{(n)} = r_j^{(n-1)} \tag{12}$$

Using Eqs. 7, 10, and 5, Eq. 9 can now be re-stated as

$$\left\{\begin{array}{c} R_i^{(n)} \\ \hline R_j^{(n)} \end{array}\right\} = \left[\begin{array}{c|c} \bar{k}_{ii}^{(n)} & \bar{k}_{ij}^{(n)} \\ \hline \bar{k}_{ji}^{(n)} & \bar{k}_{jj}^{(n)} \end{array}\right] \left\{\begin{array}{c} r_i^{(n)} \\ \hline r_j^{(n)} \end{array}\right\} \tag{13}$$

and with the aid of Eqs. 11 and 12, one finally can obtain the following recursion formula:

$$R_n = \bar{k}_{ji}^{(n-1)} r_{n-1} + (\bar{k}_{ii}^{(n)} + \bar{k}_{jj}^{(n-1)}) r_n + \bar{k}_{ij}^{(n)} r_{n+1} \tag{14}$$

Using this relation, the structural stiffness K for the whole assemblage may be stated as a tridiagonal matrix as

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ \cdot \\ \cdot \\ R_{n+1} \end{Bmatrix} = \begin{bmatrix} (\bar{k}^{(0)} + \bar{k}_{ii}^{(1)}) & \bar{k}_{ij}^{(1)} & \cdot & \cdot & \cdot & \cdot \\ \bar{k}_{ji}^{(1)} & (\bar{k}_{jj}^{(1)} + \bar{k}_{ii}^{(2)}) & \bar{k}_{ij}^{(2)} & \cdot & \cdot & \cdot \\ \cdot & \bar{k}_{ji}^{(2)} & (\bar{k}_{jj}^{(2)} + \bar{k}_{ii}^{(3)}) & \bar{k}_{ij}^{(3)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & (\bar{k}_{jj}^{(n-1)} + \bar{k}_{ii}^{(n)}) & \bar{k}_{ij}^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \bar{k}_{ji}^{(n)} & \bar{k}_{jj}^{(n)} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ \cdot \\ \cdot \\ r_{n+1} \end{Bmatrix} \qquad (15)$$

This matrix equation is the basic relation $R = Kr$ for the displacement method of analysis written in a special manner. The computational procedure stated by Eq. 15, has the advantage that $\bar{k}_{ii}$, $\bar{k}_{ij}$, $\bar{k}_{ji}$, $\bar{k}_{jj}$ for each element are developed and put into their assigned position in a do-loop of the computer program, i.e., the K matrix is developed directly from the matrices of individual elements. This eliminates the necessity of storage of the large matrices A, B, and k. The final K matrix is the only large matrix which remains in the computer. If this K matrix is stored in a conventional rectangular form, the 7090 computer can handle problems involving 45 elements. By using subroutines of skew storage and solution or by applying a recursion process[7] problems involving several hundred elements can be solved.

In a general problem neither all terms of R nor all terms of r are completely specified. Equation 15, however, can be re-stated in partitioned form as

$$\begin{Bmatrix} R \\ \hline X \end{Bmatrix} = \left[\begin{array}{c|c} k_{11} & k_{12} \\ \hline k_{21} & k_{22} \end{array}\right] \begin{Bmatrix} U \\ \hline r \end{Bmatrix} \qquad (16)$$

where the unknown displacement U can be found first, and then the unknown joint forces X . With this information, using Eq.7, all components of $S^{(n)}$ for each element of the assemblage can be determined.

After the stress-resultants $M_i^{(n)}$, $M_j^{(n)}$, $H_i^{(n)}$, $H_j^{(n)}$, and $N_j^{(n)}$ comprising $S^{(n)}$, acting on the edges of a conical element are known, the internal stresses and deformation at any point in the cone can be found using equations such as given by Flügge[1]. The constants A_1, A_2, B_1, B_2 of the Flügge equation (Ref. 1, p. 373) for the bending stresses can be determined from the previously given quantities using the following relationship:

$$\begin{Bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} \begin{Bmatrix} H_i^{(n)} \\ M_i^{(n)} \\ H_j^{(n)} \\ M_j^{(n)} \end{Bmatrix} \tag{17}$$

In the notation of the reference, this must be supplemented by $N_s = N_j \, s_j / s$ and $N_\Theta = 0$ to account for the membrane stresses.

Cylindrical Element

In the formulation of some problems cylindrical elements are encountered and their structural properties must be known. The force transformation matrix A and the displacement transformation matrix B for the cylindrical element follow directly from the previously stated Eqs. 4 and 6 upon setting $\alpha = 90^\circ$, with $\bar{r} = 1$. The flexibility of a cylindrical element, however, must be developed anew. The expressions sought can be found from the bending and membrane solutions of cylindrical shells. After some manipulation, the bending and the membrane solutions[1], corresponding to the system of edge forces shown in Fig. 4, yields the following results: *

$$\chi_i = \frac{K}{a}\Big\{ (C_{11}-C_{21}-C_{31}-C_{41})M_i + (C_{12}-C_{22}-C_{32}-C_{42})M_j + (C_{13}-C_{23}-C_{33}-C_{43})H_i + (C_{14}-C_{24}-C_{34}-C_{44})H_j \Big\}$$

$$\begin{aligned}\chi_j = \frac{K}{a}\Big\{ & \Big[-e^{-\omega}(\cos\omega + \sin\omega)C_{11} + e^{-\omega}(\cos\omega - \sin\omega)C_{21} \\ & + e^{\omega}(\cos\omega - \sin\omega)C_{31} + e^{\omega}(\cos\omega + \sin\omega)C_{41}\Big] M_i \\ & + \Big[-e^{-\omega}(\cos\omega + \sin\omega)C_{12} + e^{-\omega}(\cos\omega - \sin\omega)C_{22} \\ & + e^{\omega}(\cos\omega - \sin\omega)C_{32} + e^{\omega}(\cos\omega + \sin\omega)C_{42}\Big] M_j \\ & + \Big[-e^{-\omega}(\cos\omega + \sin\omega)C_{13} + e^{-\omega}(\cos\omega - \sin\omega)C_{23} \\ & + e^{\omega}(\cos\omega - \sin\omega)C_{33} + e^{\omega}(\cos\omega + \sin\omega)C_{43}\Big] H_i \\ & + \Big[-e^{-\omega}(\cos\omega + \sin\omega)C_{14} + e^{-\omega}(\cos\omega - \sin\omega)C_{24} \\ & + e^{\omega}(\cos\omega - \sin\omega)C_{34} + e^{\omega}(\cos\omega + \sin\omega)C_{44}\Big] H_j \Big\}\end{aligned}$$

$$\begin{aligned}\delta_i = {} & (-C_{11}-C_{31})M_i + (-C_{12}-C_{32})M_j + (-C_{13}-C_{33})H_i + (-C_{14}-C_{34})H_j \\ & + (\nu a/Et)N\end{aligned}$$

$$\begin{aligned}\delta_j = {} & (e^{-\omega}\cos\omega\, C_{11} + e^{-\omega}\sin\omega\, C_{21} + e^{\omega}\cos\omega\, C_{31} + e^{\omega}\sin\omega\, C_{41})M_i \\ & + (e^{-\omega}\cos\omega\, C_{12} + e^{-\omega}\sin\omega\, C_{22} + e^{\omega}\cos\omega\, C_{32} + e^{\omega}\sin\omega\, C_{42})M_j \\ & + (e^{-\omega}\cos\omega\, C_{13} + e^{-\omega}\sin\omega\, C_{23} + e^{\omega}\cos\omega\, C_{33} + e^{\omega}\sin\omega\, C_{43})H_i \\ & + (e^{-\omega}\cos\omega\, C_{14} + e^{-\omega}\sin\omega\, C_{24} + e^{\omega}\cos\omega\, C_{34} + e^{\omega}\sin\omega\, C_{44})H_j \\ & - (\nu a/Et)N\end{aligned}$$

---

*For simplicity, superscripts (n) on all displacements and stress-resultants are omitted.

$$e = \frac{\nu}{2\kappa}\left\{\left[e^{-\omega}(\cos\omega - \sin\omega)-1\right]C_{13} + \left[e^{-\omega}(\cos\omega + \sin\omega)-1\right]C_{23}\right.$$
$$\left. - \left[e^{\omega}(\cos\omega + \sin\omega)-1\right]C_{33} + \left[e^{\omega}(\cos\omega - \sin\omega)-1\right]C_{43}\right\}H_i$$
$$+ \frac{\nu}{2\kappa}\left\{\left[e^{-\omega}(\cos\omega - \sin\omega)-1\right]C_{14} + \left[e^{-\omega}(\cos\omega + \sin\omega)-1\right]C_{24}\right.$$
$$\left. - \left[e^{\omega}(\cos\omega + \sin\omega)-1\right]C_{34} + \left[e^{\omega}(\cos\omega - \sin\omega)-1\right]C_{44}\right\}H_j$$
$$+ (\ell/Et)N$$

where $\omega = \kappa\frac{\ell}{a}$, C_{11}, C_{12} .... C_{43}, C_{44} satisfy the relations

$$\begin{aligned}
C_1 &= C_{11}M_i + C_{12}M_j + C_{13}H_i + C_{14}H_j \\
C_2 &= C_{21}M_i + C_{22}M_j + C_{23}H_i + C_{24}H_j \\
C_3 &= C_{31}M_i + C_{32}M_j + C_{33}H_i + C_{34}H_j \\
C_4 &= C_{41}M_i + C_{42}M_j + C_{43}H_i + C_{44}H_j
\end{aligned} \tag{18}$$

$C_{11} = D_{11}a^2 /(\Delta\ 2K\kappa^2)$ $\qquad$ $C_{12} = -D_{21}a^2 / (\Delta\ 2K\kappa^2)$

$C_{13} = D_{31}a^3 / (\Delta\ 2K\kappa^3)$ $\qquad$ $C_{14} = -D_{41}a^3 / (\Delta\ 2K\kappa^3)$

$C_{21} = -D_{12}a^2 / (\Delta\ 2K\kappa^2)$ $\qquad$ $C_{22} = D_{22}a^2 / (\Delta\ 2K\kappa^2)$

$C_{23} = -D_{32}a^3 / (\Delta\ 2K\kappa^3)$ $\qquad$ $C_{24} = D_{42}a^3 / (\Delta\ 2K\kappa^3)$

$C_{31} = D_{13}a^2 / (\Delta\ 2K\kappa^2)$ $\qquad$ $C_{32} = -D_{23}a^2 / (\Delta\ 2K\kappa^2)$

$C_{33} = D_{33}a^3 / (\Delta\ 2K\kappa^3)$ $\qquad$ $C_{34} = -D_{43}a^3 / (\Delta\ 2K\kappa^3)$

$C_{41} = -D_{14}a^2 / (\Delta\ 2K\kappa^2)$ $\qquad$ $C_{42} = D_{24}a^2 / (\Delta\ 2K\kappa^2)$

$C_{43} = -D_{34}a^3 / (\Delta\ 2K\kappa^3)$ $\qquad$ $C_{44} = D_{44}a^3 / (\Delta\ 2K\kappa^3)$

$$\Delta = \left[2\sin\omega + (e^{\omega} - e^{-\omega})\right]\left[2\sin\omega - (e^{\omega} - e^{-\omega})\right]$$

$$D_{11} = \cos 2\,\omega - \sin 2\,\omega - e^{2\omega}$$

$$D_{21} = -(\cos\omega + \sin\omega)(e^{\omega} - e^{-\omega}) - 2\sin\omega\, e^{\omega}$$

$$D_{31} = e^{2\omega} - 1 - \sin 2\omega$$

$$D_{41} = \cos\omega\,(e^{\omega} - e^{-\omega}) - 2\sin\omega\, e^{\omega}$$

$$D_{12} = (\cos\omega - \sin\omega)^2 + 2\sin^2\omega - e^{2\omega}$$

$$D_{22} = -\cos\omega\,(e^{\omega} - e^{-\omega}) - \sin\omega\,(e^{\omega} + e^{-\omega})$$

$$D_{32} = 2\sin^2\omega$$

$$D_{42} = -\sin\omega\,(e^{\omega} - e^{-\omega})$$

$$D_{13} = -e^{-2\omega} + \sin 2\omega + \cos 2\omega$$

$$D_{23} = (\cos\omega - \sin\omega)(e^{\omega} - e^{-\omega}) + 2\sin\omega\, e^{-\omega}$$

$$D_{33} = 1 - e^{-2\omega} - \sin 2\omega$$

$$D_{43} = \cos\omega\,(e^{\omega} - e^{-\omega}) - 2\sin\omega\, e^{-\omega}$$

$$D_{14} = e^{-2\omega} - 2\sin^2\omega - \sin 2\omega - 1$$

$$D_{24} = -\cos\omega\,(e^{\omega} - e^{-\omega}) - \sin\omega\,(e^{\omega} + e^{-\omega})$$

$$D_{34} = 2\sin^2\omega$$

$$D_{44} = -\sin\omega\,(e^{\omega} - e^{-\omega})$$

From the above equation, the flexibility matrix analogous to that given schematically by Eq. 1 can be obtained. In this matrix the elements $f_{15}^{(n)}$, $f_{25}^{(n)}$, $f_{51}^{(n)}$, $f_{52}^{(n)}$ are zero, and $N_j^{(n)} = N^{(n)}$. By inverting the f matrix, the stiffness matrix k for a cylindrical ring is obtained.

In a solution of a complete problem, after the edge stress-resultants $M_i^{(n)}$, $M_j^{(n)}$, $H_i^{(n)}$, $H_j^{(n)}$, $N^{(n)}$ for a ring are determined, stresses and displacements at any point of the ring can be found. For the equations such as given by Flügge (Ref. 1, p. 228) the constants C_1, C_2, C_3, C_4 are given by Eq. 18 of this paper.

Spherical Cap

In many instances a shell of revolution may be closed at an end. In finite element solution this closure can be approximated by a shallow spherical cap. The required structural properties for such a cap are found by first comparing the displacements and the force systems shown in Fig. 3 and 5. From this, the displacement transformation matrix $B^{(0)}$ and the force transformation matrix $A^{(0)}$ may be formulated as follows:

$$\begin{Bmatrix} \chi^{(0)} \\ \delta_h^{(0)} \\ \delta_v^{(0)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \theta_1^{(0)} \\ \Delta h_1^{(0)} \\ \Delta v_1^{(0)} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} T^{(1)} \\ P_h^{(1)} \\ P_v^{(1)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} M^{(0)} \\ H^{(0)} \\ Q^{(0)} \end{Bmatrix} \tag{19}$$

or in symbolic form corresponding to the analogous Eqs. 5 and 7,

$$v^{(0)} = B^{(0)} r^{(0)} \qquad \text{and} \qquad R^{(0)} = A^{(0)} S^{(0)} \tag{20}$$

No subscripts for the symbols in the last equation are necessary since there are no upper and lower edges in a cap.

The force-deformation relationship for the cap must be established next. Such an equation, using the flexibility matrix, can be written as

$$\begin{Bmatrix} \chi^{(0)} \\ \delta h^{(0)} \\ \delta v^{(0)} \end{Bmatrix} = \begin{bmatrix} f_{11}^{(0)} & f_{12}^{(0)} & f_{13}^{(0)} \\ f_{21}^{(0)} & f_{22}^{(0)} & f_{23}^{(0)} \\ f_{31}^{(0)} & f_{32}^{(0)} & f_{33}^{(0)} \end{bmatrix} \begin{Bmatrix} M^{(0)} \\ H^{(0)} \\ Q^{(0)} \end{Bmatrix}$$

or in the abbreviated notation as $v^{(0)} = f^{(0)} S^{(0)}$. The inverse of $f^{(0)}$ is the stiffness matrix $k^{(0)}$ of the cap.

To determine the elements of the $f^{(0)}$ matrix, the available solution(1)(2) for bending of a shallow spherical shell was utilized. For this purpose, the shallow cap was assumed supported by a concentrated force P as shown in Fig. 5, and the change in vertical height $\delta v^{(0)}$ was determined corresponding to balanced vertical forces. This approximate solution causes a local "hump" near the singular point. By making the cap, however, sufficiently small, the effect of this local phenomenon can be minimized. Other deformations were found in a conventional manner. On this basis, the following expressions apply:

$$\begin{aligned}
\chi^{(0)} = {} & (1/\ell)(C_{11}\operatorname{ber}'x_o + C_{21}\operatorname{bei}'x_o)M^{(0)} \\
& + (1/\ell)(C_{12}\operatorname{ber}'x_o + C_{22}\operatorname{bei}'x_o)H^{(0)} \\
& + (1/\ell)(C_{13}\operatorname{ber}'x_o + C_{23}\operatorname{kei}'x_o - (\ell^2 r_o/K)\operatorname{kei}'x_o)Q^{(0)}
\end{aligned}$$

$$\begin{aligned}
\delta h^{(0)} = {} & (\ell/a)\Big\{C_{11}\Big[x_o \operatorname{ber} x_o - (1+\nu)\operatorname{bei}'x_o\Big] \\
& + C_{21}\Big[x_o \operatorname{bei} x_o + (1+\nu)\operatorname{ber}'x_o\Big]\Big\} M^{(0)} \\
& + (\ell/a)\Big\{C_{12}\Big[x_o \operatorname{ber} x_o - (1+\nu)\operatorname{bei}'x_o\Big] \\
& + C_{22}\Big[x_o \operatorname{bei} x_o + (1+\nu)\operatorname{ber}'x_o\Big]\Big\} H^{(0)} \\
& + \Big\{(\ell/a)C_{13}\Big[x_o \operatorname{ber} x_o - (1+\nu)\operatorname{bei}'x_o\Big] \\
& + (\ell/a)C_{23}\Big[x_o \operatorname{bei} x_o + (1+\nu)\operatorname{ber}'x_o\Big] \\
& - (\ell^3 r_o/aK)\Big[x_o \operatorname{kei} x_o + (1+\nu)\operatorname{ker}'x_o + (1+\nu)/x_o\Big]\Big\} Q^{(0)}
\end{aligned}$$

$$\delta_v^{(0)} = \Big\{ C_{11}\Big[-(\ell^2/a^2)x_o\,(1+\nu)\ \mathrm{bei}'x_o - \mathrm{ber}\ x_o\Big]$$
$$+ C_{21}\Big[(\ell^2 x_o/a^2)(1+\nu)\ \mathrm{ber}'x_o - \mathrm{bei}\ x_o\Big]\Big\}\, M^{(0)}$$
$$+ \Big\{ C_{12}\Big[-(\ell^2 x_o/a^2)(1+\nu)\ \mathrm{bei}'x_o - \mathrm{ber}\ x_o\Big]$$
$$+ C_{22}\Big[(\ell^2 x_o/a^2)(1+\nu)\ \mathrm{ber}'x_o - \mathrm{bei}\ x_o\Big]\Big\}\, H^{(0)}$$
$$+ \Big\{ C_{13}\Big[-(\ell^2 x_o/a^2)(1+\nu)\ \mathrm{bei}'x_o - \mathrm{ber}\ x_o\Big]$$
$$+ C_{23}\Big[(\ell^2 x_o/a^2)(1+\nu)\ \mathrm{ber}'x_o - \mathrm{bei}\ x_o\Big]$$
$$- (\ell^2 r_o/K)\Big[(\ell^2 x_o/a^2)(1+\nu)(\mathrm{ker}'x_o + 1/x_o) - \mathrm{kei}\ x_o\Big]\Big\}\, Q^{(0)}$$

where $C_{11} = -(\ell^2/\Delta K)\ \mathrm{ber}'x_o$

$$C_{12} = -(1/\Delta Et)(a^3 x_o/(a^2 + r_o^2))\Big[\mathrm{ber}\ x_o - (1-\nu)\mathrm{bei}'x_o/x_o\Big] \qquad (21)$$

$$C_{13} = (1/\Delta)\Big\{(\ell^2 r_o/K)\Big[\mathrm{ber}\ x_o\ \mathrm{ker}'x_o - \mathrm{ber}'x_o\ \mathrm{ker}\ x_o$$
$$- (1-\nu)\mathrm{bei}'x_o\ \mathrm{ker}'x_o/x_o + (1-\nu)\mathrm{ber}'x_o\ \mathrm{ker}'x_o/x_o\Big]$$
$$+ (\ell^3/K)(a^2/(a^2 + r_o^2))\Big[\mathrm{ber}\ x_o - (1-\nu)\mathrm{bei}'x_o/x_o\Big]\Big\}$$

$$C_{21} = -(\ell^2/\Delta K)\ \mathrm{bei}'x_o$$

$$C_{22} = -(1/\Delta Et)(a^3 x_o/(a^2 + r_o^2))\Big[\mathrm{bei}\ x_o + (1-\nu)\mathrm{ber}'x_o/x_o\Big]$$

$$C_{23} = (1/\Delta)\ \ (\ell^2 r_o/K)\Big[\mathrm{bei}\ x_o\ \mathrm{ker}'x_o - \mathrm{bei}'x_o\ \mathrm{ker}\ x_o$$
$$+ (1-\nu)\mathrm{ber}'x_o\ \mathrm{ker}'x_o/x_o + (1-\nu)\mathrm{bei}'x_o\ \mathrm{kei}'x_o/x_o\Big]$$
$$+ (\ell^3/K)(a^2/(a^2 + r_o^2))\Big[\mathrm{bei}\ x_o + (1-\nu)\mathrm{ber}'x_o/x_o\Big]$$

$$x_o = r_o/\ell$$

$$\ell = (at)^{\frac{1}{2}}/\Big[12(1-\nu^2)\Big]^{\frac{1}{4}}$$

and
$$\Delta = \begin{vmatrix} -\Big[\mathrm{bei}\ x_o + (1-\nu)\mathrm{ber}'x_o/x_o\Big] & \Big[\mathrm{ber}\ x_o - (1-\nu)\mathrm{bei}'x_o/x_o\Big] \\ \mathrm{bei}'x_o & -\mathrm{ber}'x_o \end{vmatrix}$$

Joint Loads

If the applied loads are concentrated and act at the joints of the assemblage, the solution procedure given above is directly applicable. For distributed loadings such as occur, for example, when a shell is pressurized, the loading condition must be approximated by concentrated forces applied at nodal points. For this purpose two schemes were developed. In the one, the distributed load is concentrated at several of the nodal points; in the other, reversed fixed-edge forces are applied at the joints. In the latter case, the fixed-edge forces must be superimposed on the element forces obtained from the solution of the assemblage of elements to give the final result.

(a) The approximate load on joint n, Fig. 3, simply taken as the components of the total pressure on the tributary (shaded) areas as shown in Fig. 6 are

$$\begin{aligned} T_n &= 0 \\ P_{hn} &= \left\{ \left[(r_a + r_n)/r_n \right] (a/2)\sin\alpha_a + \left[(r_n + r_b)/r_n \right] (b/2)\sin\alpha_b \right\} p_r \\ P_{vn} &= \left\{ \left[(r_a + r_n)/r_n \right] (a/2)\cos\alpha_a + \left[(r_n + r_b)/r_n \right] (b/2)\cos\alpha_b \right\} p_r \end{aligned} \tag{22}$$

where r_a , r_b are two radii of the centroidal circles for element a above and b below the joint n , and α_a , α_b are the inclinations of the respective elements. For such an approximation of loading, the loads contributed by the pressure on the spherical cap are assumed concentrated along the bottom edge of the cap. p_r is the internal pressure.

(b) The joint loads from fixed-edge forces can be found from the following expression

$$R^{(n)} = -A^{(n)} S_F^{(n)} = A^{(n)} k^{(n)} v_m^{(n)} \tag{23}$$

where $S_F^{(n)}$ is the matrix representing the fixed-edge forces acting on an element, and $v_m^{(n)}$ is the matrix for the membrane deformations of the element. For future reference, the membrane displacements of the elements are listed below. For the truncated cone having no meridional displacement of the top edge, these relations are

$$
\begin{aligned}
N_{mi}^{(n)} &= (p_r \cot\alpha \,/\, 2) \left[s_i - (s_j^2/s_i) \right] \\
\chi_{mi}^{(n)} &= -(p_r \cot^2\alpha \,/\, 2Et) \left[3\, s_i + (s_j^2/s_i) \right] \\
\chi_{mj}^{(n)} &= (p_r \cot^2\alpha \,/\, Et)\; 2\, s_j \\
\delta_{mi}^{(n)} &= -(p_r \cos^2\alpha/(2Et \sin\alpha)) \left[(2-\nu) s_i^2 + \nu\, s_j^2 \right] \\
\delta_{mj}^{(n)} &= (p_r \cos^2\alpha \,/(Et \sin\alpha)) s_j^2 \\
e_m^{(n)} &= (p_r \cot\alpha/\, 4Et) \left[(1-2\nu)(s_j^2 - s_i^2) - 2\, s_j^2 \log\, (s_j/s_i) \right]
\end{aligned}
\tag{24}
$$

The corresponding expressions for cylindrical element are

$$
\begin{aligned}
\chi_{mi}^{(n)} &= 0 & \qquad \chi_{mj}^{(n)} &= 0 \\
\delta_{mi}^{(n)} &= -p_r\, a^2/\, Et & \qquad \delta_{mj}^{(n)} &= p_r\, a^2/\, Et \\
e_m^{(n)} &= -\nu\, p_r\, a\ell/\, Et & &
\end{aligned}
\tag{25}
$$

and for the spherical cap are

$$
\begin{aligned}
\chi_m^{(0)} &= (1-\nu)\, p_r\, r_o \,/\, 2Et \\
\delta_{hm}^{(0)} &= (1-\nu)\, p_r\, r_o\, a \,/\, 2Et \\
\delta_{hm}^{(0)} &= (1-\nu)\, p_r\, r_o^2 \,/\, 2Et
\end{aligned}
\tag{26}
$$

Illustrative Example

As an application of the procedures described above, results of several analyses of a spherical shell are reported. A spherical shell was selected for ease of comparison with known solutions. The shell considered is 0.5 in. thick, has a 100 in. radius, and a central semi-angle of 75°. The edges are clamped, and the unit is pressurized to 100 psi. Let $E = 10^7$ psi, and $\nu = 0.2$.

This shell was analyzed by three different procedures. In the one, reversed fixed-end forces were applied at the joints of the assemblage and the problem solved. The resulting meridional moments for three different arrangements of elements are shown in Fig. 7. The angles subtended by the elements ranging from top to bottom are noted on the figure. Alternatively, the problem was re-solved using approximate joint loads. The end results are shown in Fig. 8. In these solutions equal size elements were used in each case. The assumed subtended angles of 1.5°, 2°, and 3°, respectively, are noted on the figure.

In examining the two sets of solutions it is seen that the stresses near the apex are unrealistic and must be discarded. The assumed cap of 5° semi-angle is too large in comparison with the conical elements. Instead of having a cap at all, however, a much larger portion of the shell may be completely removed, and only a force determined from the membrane analysis applied at the newly created boundary. This procedure minimizes the number of elements required in the analysis and is recommended whenever applicable. Also from the diagrams it can be noted, with particular clarity from Fig. 8 where the same size elements are used, that over a considerable range of ϕ the curves are horizontal. This range in the real structure corresponds to the zone where membrane stresses are predominant and where virtually no bending stresses occur. The structure was analyzed, however, as if it were loaded with concentrated forces. This distorts the real situation and indicates the presence of moments of constant magnitude over a large zone of the shell. By taking the length of elements very small, these "residual moments" can be reduced to negligible amounts. It also appears sufficiently justified to simply shift the base-line of the graph by the amount equal to the residual moment corresponding to the size of the elements provided the element lengths and break angles are constant. On this basis, for example, the corrected maximum moments for the three sizes of elements are 595.9, 596.0, and 597.8, respectively. These values compare very favorably with the value of 589.2 which is obtained from Hetenyi's equations for a spherical shell[8].

Excellent results are obtained with a minimum number of elements if edge distortions in the primary structure calculated on the assumption of membrane action are made compatible with the prescribed boundary conditions using the finite element model. This does not introduce in the interior joints any "residual moments" nor any joint loads. Solutions of the same problem based on this procedure are shown in Fig. 9.

Based on the above and other examples, it is concluded that the proposed method can yield sufficiently accurate results for most practical purposes. Moreover, the method is highly versatile and can be applied to problems which cannot be solved by any other means.

Note:

After the work on this phase of the investigation was completed, a paper using somewhat similar approach by Peter E. Grafton and Donald R. Strome on "Analysis of Axisymmetrical Shells by the Direct Stiffness Method" appeared in the October 1963 AIAA Journal, Vol. 1, No. 10, pp. 2342-2347. In this paper several aspects of the solution are done differently.

LIST OF FIGURES

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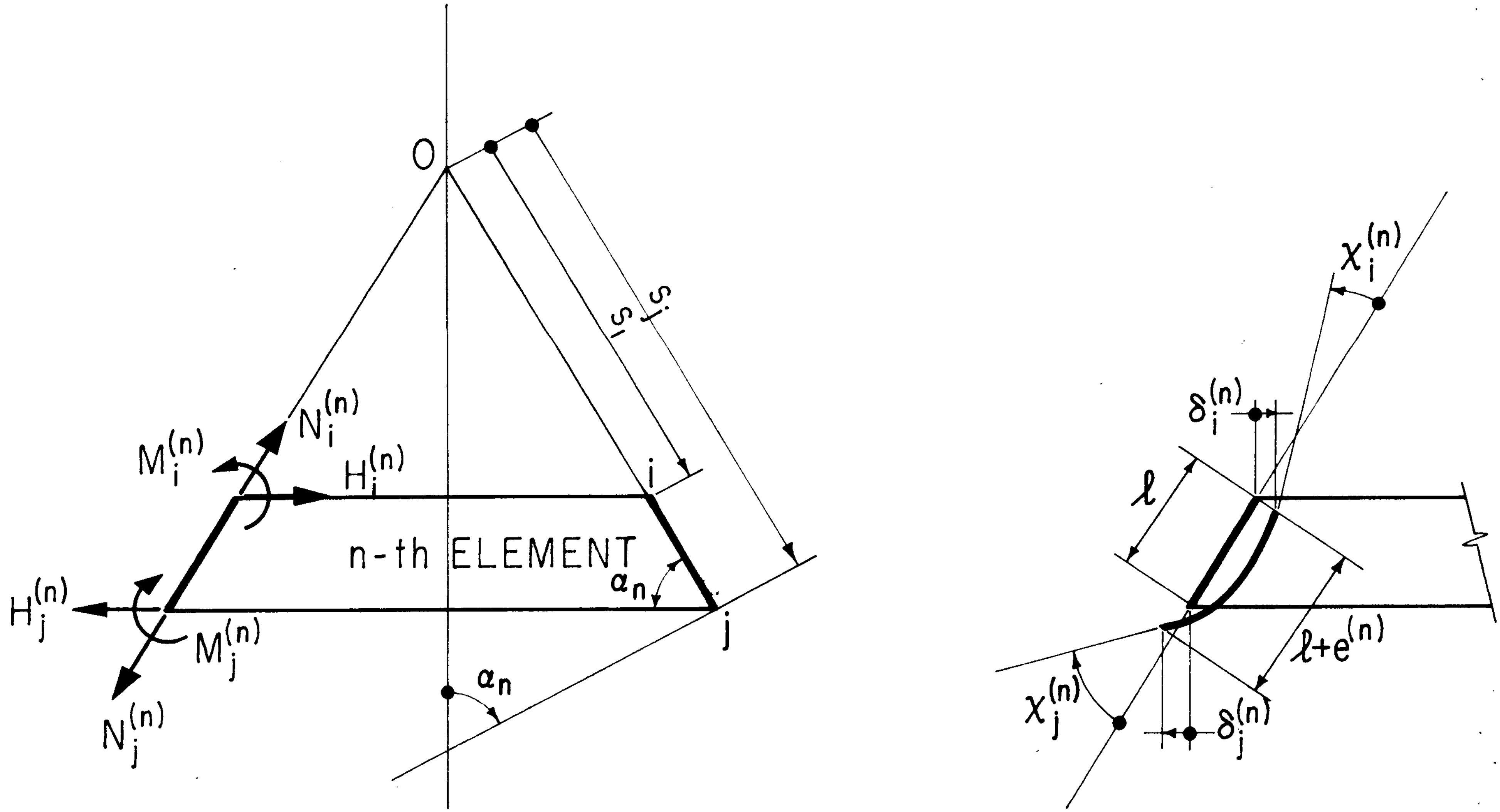

O
s_i
s_j
i
j
$N_i^{(n)}$
$M_i^{(n)}$
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n-th ELEMENT
α_n
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$N_j^{(n)}$
α_n
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$\ell+e^{(n)}$
$\chi_j^{(n)}$
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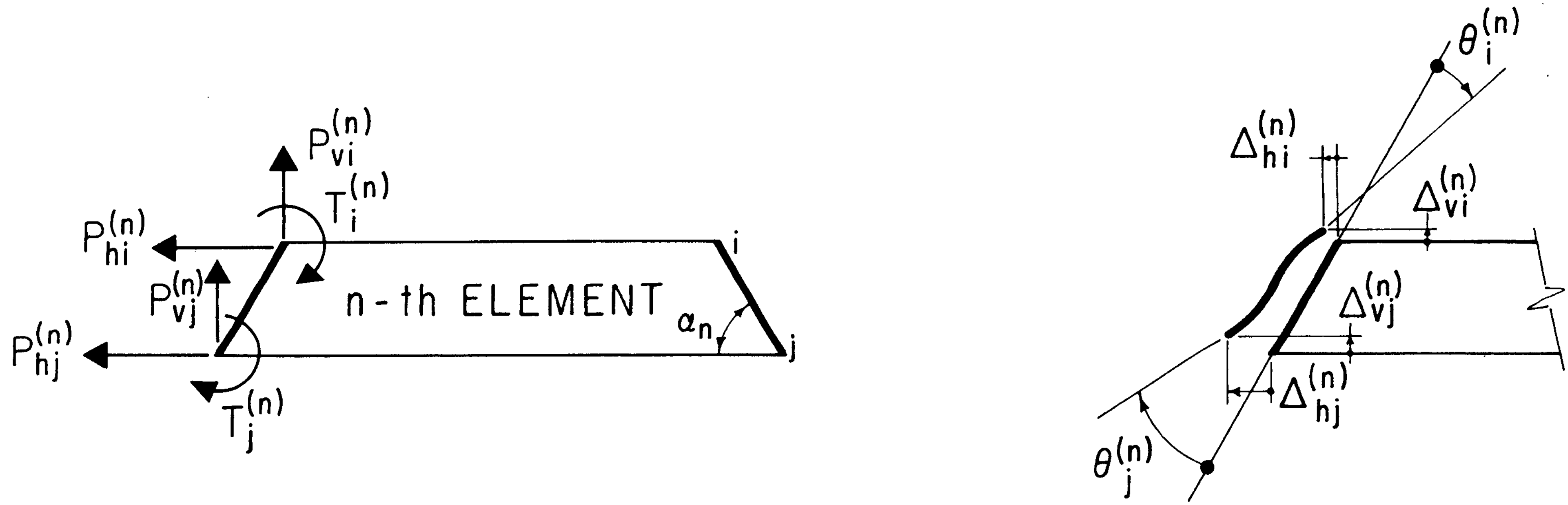
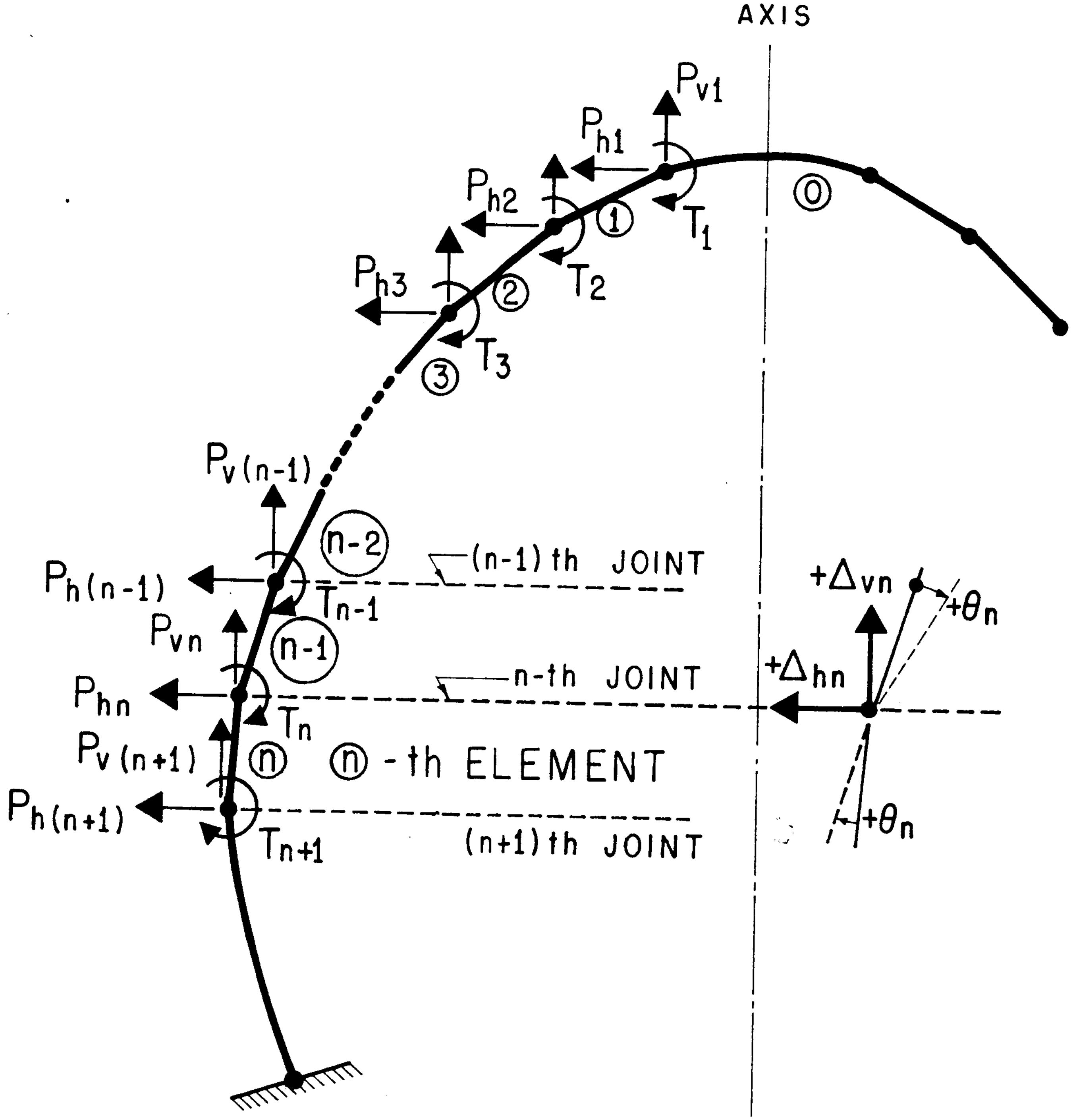
FIG. 1



$P_{vi}^{(n)}$
$T_i^{(n)}$
$P_{hi}^{(n)}$
$P_{vj}^{(n)}$
$P_{hj}^{(n)}$
$T_j^{(n)}$
n-th ELEMENT
α_n
i
j
$\theta_i^{(n)}$
$\Delta_{hi}^{(n)}$
$\Delta_{vi}^{(n)}$
$\Delta_{vj}^{(n)}$
$\Delta_{hj}^{(n)}$
$\theta_j^{(n)}$

FIG. 2



AXIS
P_{v1}
P_{h1}
P_{h2}
P_{h3}
T_1
T_2
T_3
0
1
2
3
$P_{v(n-1)}$
n-2
(n-1) th JOINT
$P_{h(n-1)}$
T_{n-1}
P_{vn}
n-1
n-th JOINT
P_{hn}
T_n
$P_{v(n+1)}$
n
n -th ELEMENT
$P_{h(n+1)}$
T_{n+1}
(n+1) th JOINT
$+\Delta_{vn}$
$+\theta_n$
$+\Delta_{hn}$
$+\theta_n$

FIG. 3

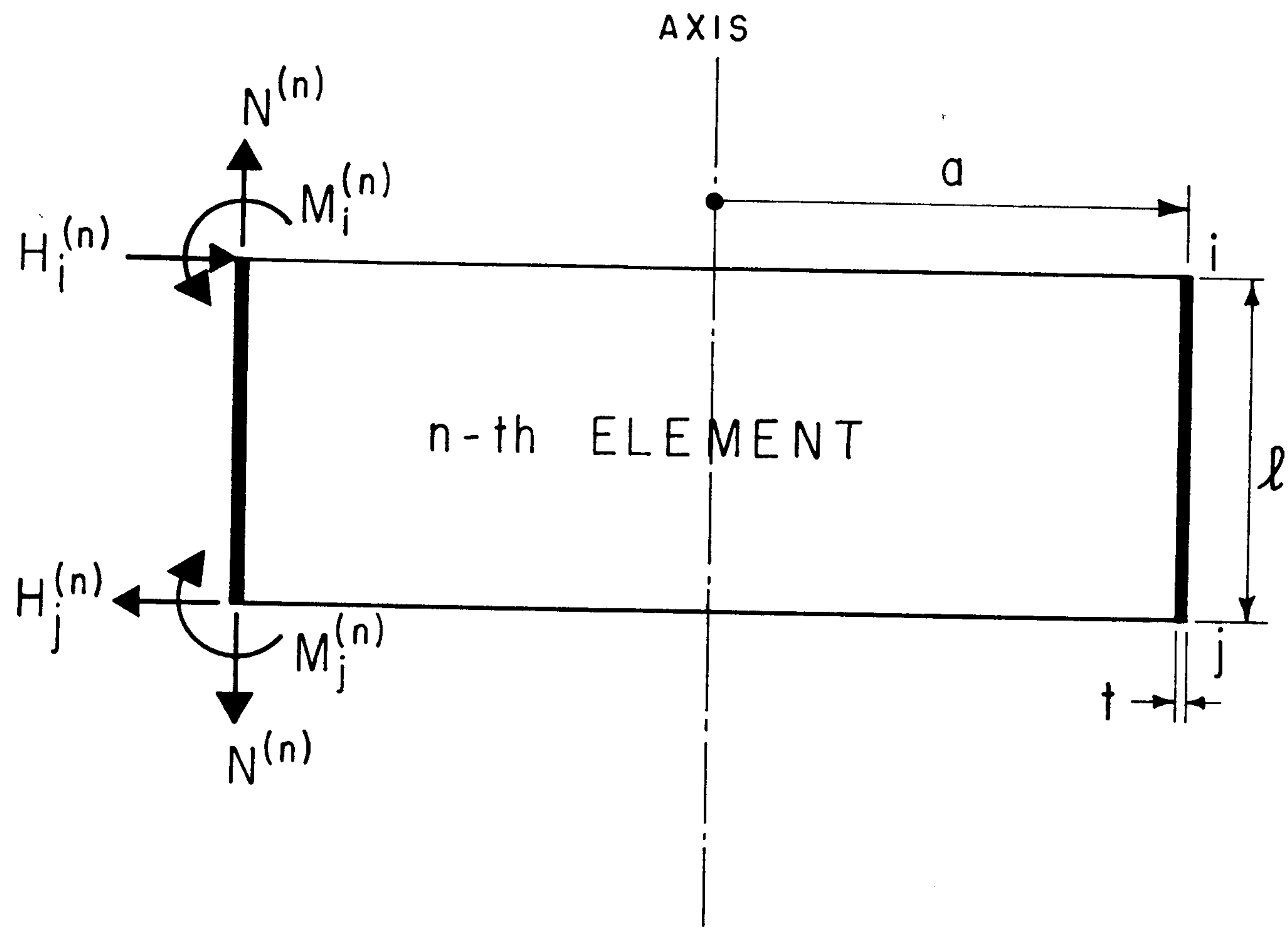


FIG. 4

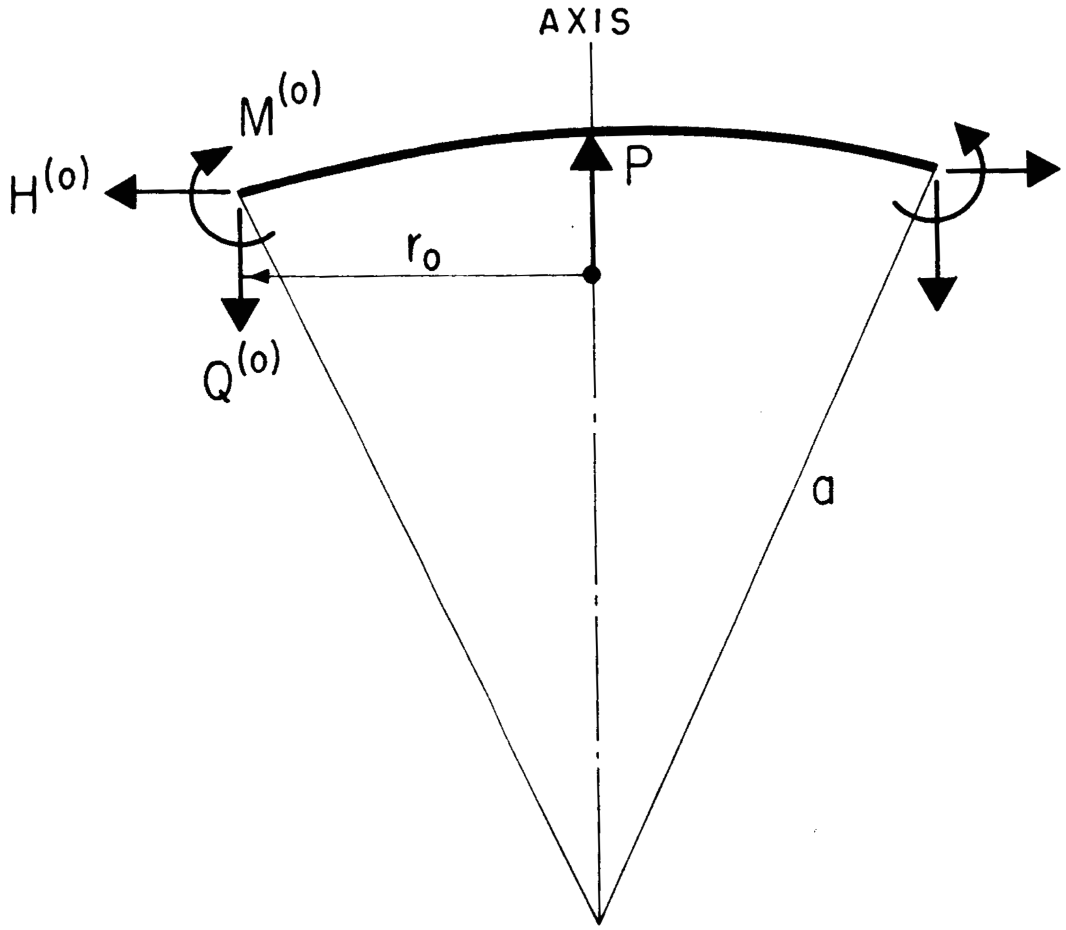


FIG. 5

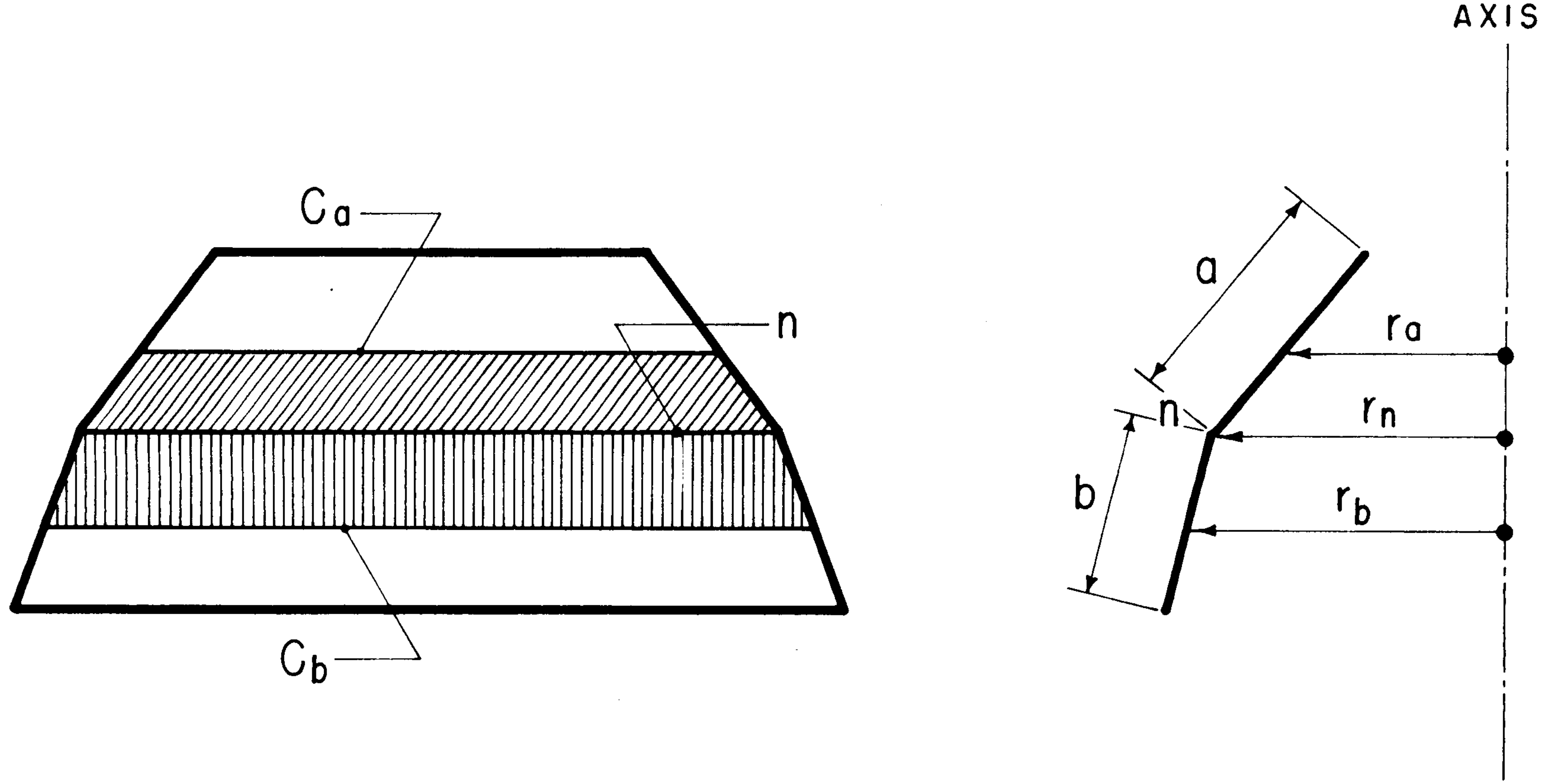
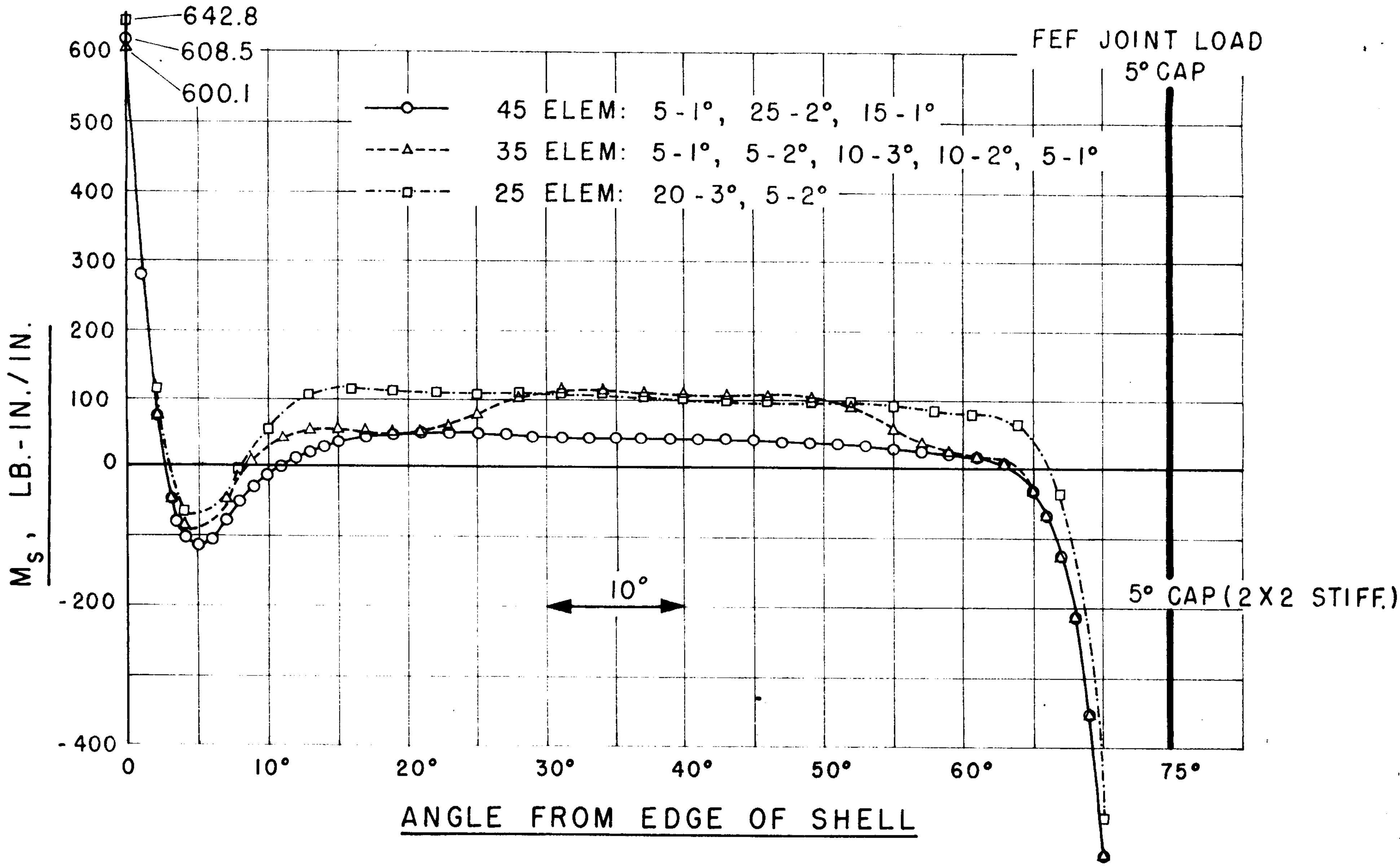
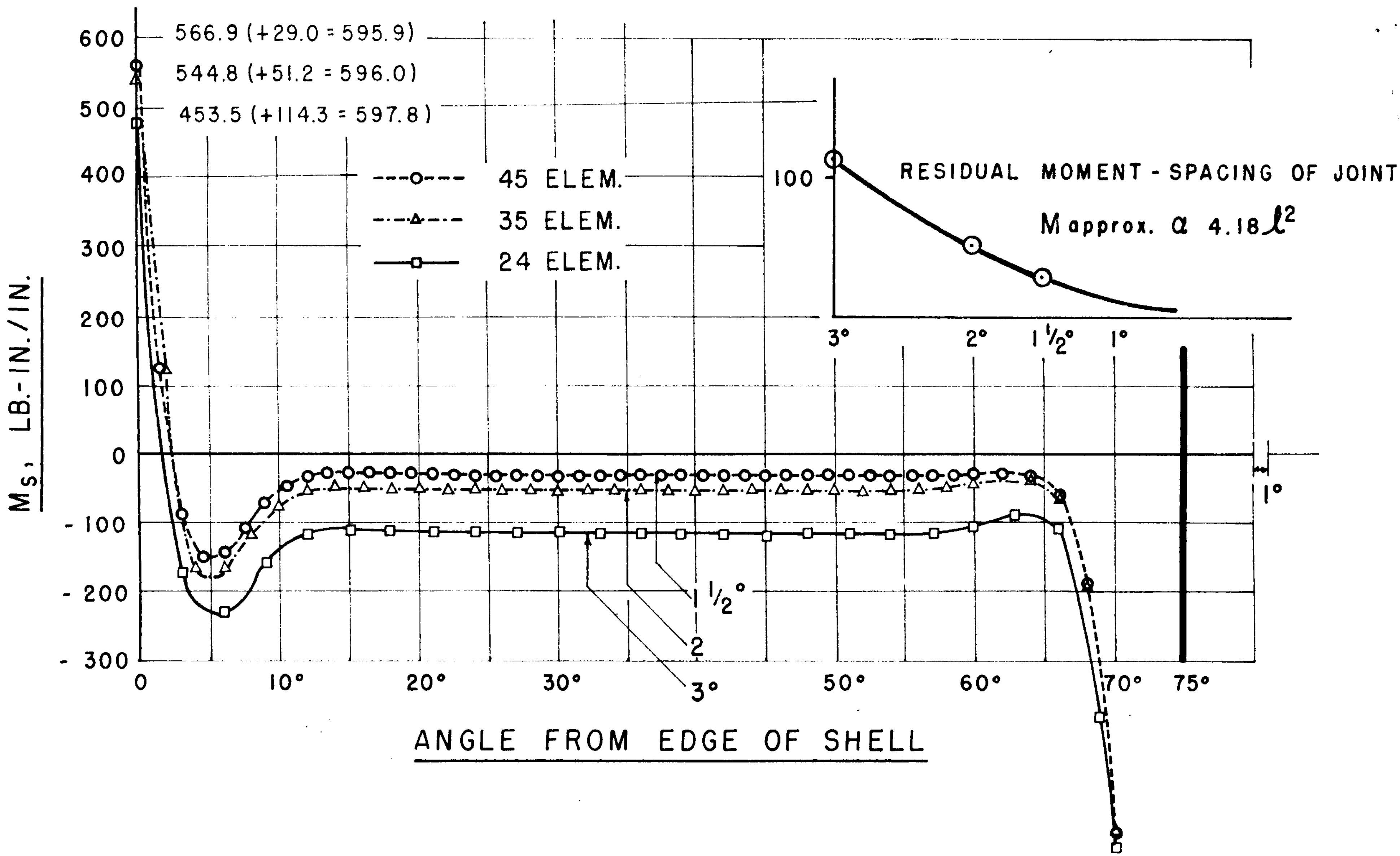


FIG. 6



642.8
608.5
600.1
FEF JOINT LOAD
5° CAP
45 ELEM: 5 - 1°, 25 - 2°, 15 - 1°
35 ELEM: 5 - 1°, 5 - 2°, 10 - 3°, 10 - 2°, 5 - 1°
25 ELEM: 20 - 3°, 5 - 2°
600
500
400
300
200
100
0
-200
-400
M_S, LB.-IN./IN.
10°
5° CAP (2 X 2 STIFF.)
0
10°
20°
30°
40°
50°
60°
75°
ANGLE FROM EDGE OF SHELL

FIG. 7



566.9 (+29.0 = 595.9)
544.8 (+51.2 = 596.0)
453.5 (+114.3 = 597.8)
45 ELEM.
35 ELEM.
24 ELEM.
RESIDUAL MOMENT - SPACING OF JOINT
M approx. α 4.18 ℓ²
100
3°
2°
1 1/2°
1°
600
500
400
300
200
100
0
-100
-200
-300
0
10°
20°
30°
50°
60°
70°
75°
1 1/2°
2
3°
M_S, LB.-IN./IN.
ANGLE FROM EDGE OF SHELL

FIG. 8

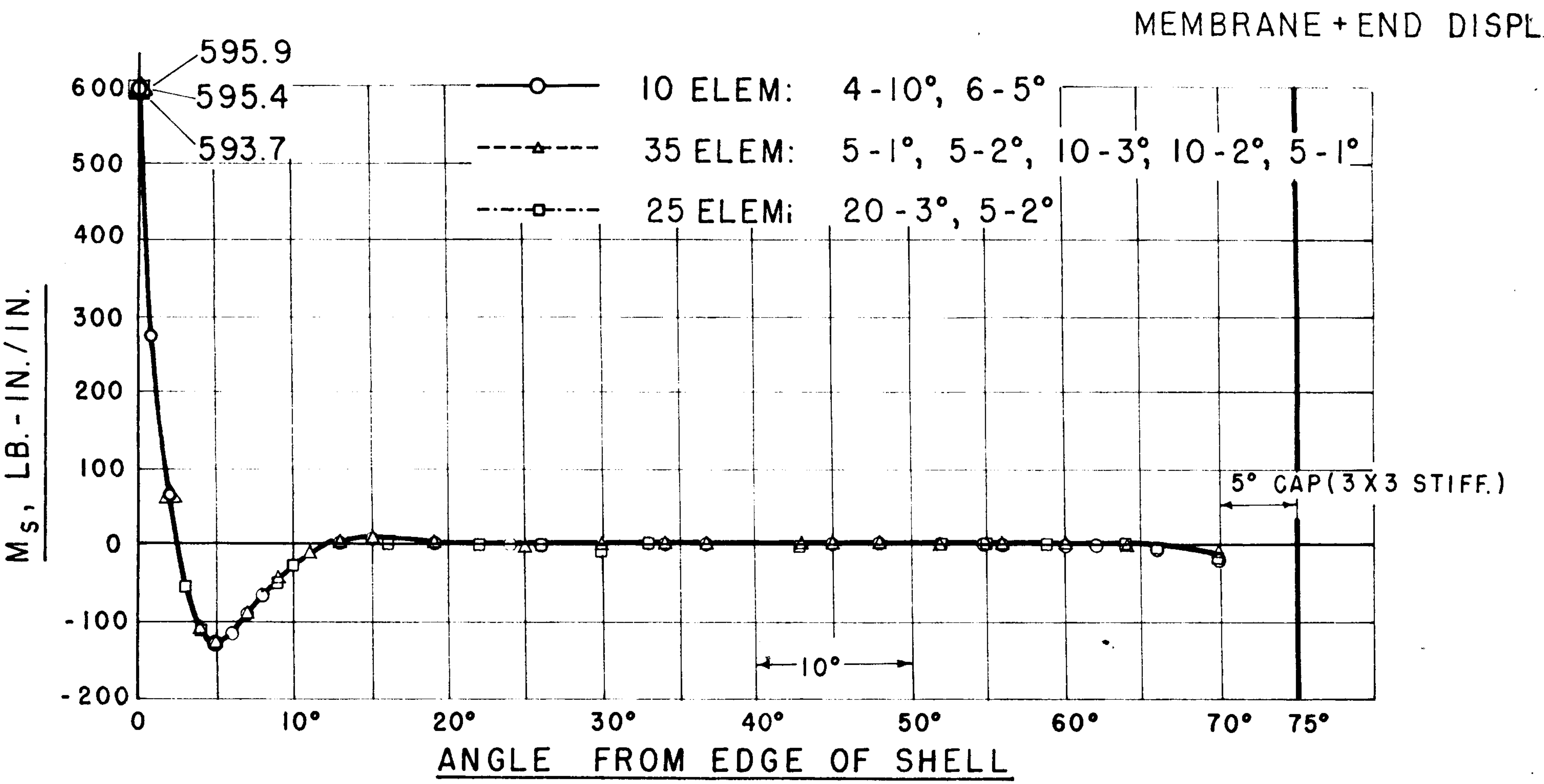

MEMBRANE + END DISPL.
595.9
595.4
593.7
10 ELEM: 4-10°, 6-5°
35 ELEM: 5-1°, 5-2°, 10-3°, 10-2°, 5-1°
25 ELEMi 20-3°, 5-2°
600
500
400
300
200
100
0
-100
-200
M S, LB.-IN./IN.
5° CAP (3 X 3 STIFF.)
10°
0
10°
20°
30°
40°
50°
60°
70°
75°
ANGLE FROM EDGE OF SHELL

FIG. 9